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Bergman–Hartogs domains and their automorphisms

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Области Бергмана–Гартогса и их автоморфизмы

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Abstract. For Cartan–Hartogs domains and also for Bergman–Hartogs domains, the determination of their automorphism groups is given for the cases when the base is any bounded symmetric domain and a general bounded homogeneous domain respectively.

Keywords: bounded symmetric domains; bounded homogeneous domain; automorphisms

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Аннотация. Для областей Картана–Гартогса, а также для областей Бергмана–Гартогса находятся их группы автоморфизмов — соответственно для случаев, когда база есть произвольная ограниченная симметрическая область и общая ограниченная однородная область.

Ключевые слова: ограниченные симметрические области; ограниченные однородные области; автоморфизмы

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Cartan–Hartogs domains (see definition below) are in general non homogeneous, but their automorphism group acts transitively on the real hypersurfaces of a one parameter family. The exact automorphism group has been determined by Ahn Heungju, Byun Jisoo, Park Jong-do [1] when the base Ω of the Cartan–Hartogs domain is a bounded symmetric domain of classical type. Their method, using the Wong–Rosay theorem, may be extended to the case where the base Ω is any bounded symmetric domain. The result holds also for

"Bergman–Hartogs domains" which are defined in the same way as Cartan–Hartogs domains, with a base Ω which is a general bounded homogeneous domain.

1. Definitions and notations

1.1. Bergman kernel. Let Ω be a bounded domain in a complex space V of dimension d. Let V be oriented by a translation invariant volume form ω. Let

$$\mathcal{H}(\Omega) = \left\{ f \in \mathcal{O}(\Omega) \mid \|f\|_\Omega^2 := \int_\Omega |f(z)|^2 \ \omega(z) < +\infty \right\}$$

be the *Bergman space* of Ω. Then $\mathcal{H}(\Omega)$, with the scalar product

$$(u \mid v)_\Omega := \int_\Omega u(z)\,\overline{v(z)}\,\omega(z)$$

is a *Hilbert space of holomorphic functions* (that is, $\mathcal{H}(\Omega)$ is a Hilbert space and the inclusion $\mathcal{H}(\Omega) \hookrightarrow \mathcal{O}(\Omega)$ is continuous). For $z \in \Omega$, let $K_{\Omega,z} \in \mathcal{H}(\Omega)$ such that

$$f(z) = (f \mid K_{\Omega,z})_\Omega$$

for all $f \in \mathcal{H}(\Omega)$. The *Bergman kernel* of Ω is the reproducing kernel

$$K(z,t) = K_\Omega(z,t) = \overline{K_{\Omega,z}(t)}$$

of $\mathcal{H}(\Omega)$. Denote

$$\mathcal{K}(z) = \mathcal{K}_\Omega(z) := K_\Omega(z,z) = \|K_{\Omega,z}\|_\Omega^2$$

(which is also called Bergman kernel of Ω).

If $g : \Omega \to \Omega$ is a holomorphic automorphism of Ω, then

$$\mathcal{K}_\Omega(gz) = \frac{\mathcal{K}_\Omega(z)}{|Jg(z)|^2},$$

where $Jg(z)$ is the complex Jacobian of g at z.

1.2. Cartan domains. Let Ω be an *irreducible complex symmetric domain of non compact type ("Cartan domain")*, realized as the spectral unit ball of a *simple Hermitian positive Jordan triple* V.

We denote by (a, b, r) the *numerical invariants* of V; by γ the *genus* of V: $\gamma = 2 + a(r-1) + b$ and by $\mathcal{N}(x,y)$ the *generic norm* of V (which is an irreducible polynomial of bidegree (r,r)).

The Bergman kernel of Ω is then

$$\mathcal{K}_\Omega(z) = \mathcal{K}_\Omega(0)\mathcal{N}(z,z)^{-\gamma}.$$

1.3. Cartan–Hartogs domains.

D e f i n i t i o n 1. For a real number $\mu > 0$ and an integer $N > 0$, let $\widetilde{\Omega}$ be the Hartogs type domain defined by

$$\widetilde{\Omega} = \widetilde{\Omega}(\mu, N) := \left\{(z, Z) \in \Omega \times \mathbb{C}^N \mid \|Z\|^2 < \mathcal{N}(z, z)^{\mu}\right\}.$$

The domain $\widetilde{\Omega}(\mu, N)$ is called *Cartan–Hartogs domain.*

Cartan–Hartogs domains have been introduced by Weiping Yin and G. Roos in 1998. They generalize various domains like *complex ellipsoids (Thullen domains).*

1.4. Bergman–Hartogs domains. Let Ω be a bounded complex domain. Let $c > 0$ be a positive real number and $N > 0$ an integer.

D e f i n i t i o n 2. The *Bergman–Hartogs domain* $\widehat{\Omega}(c, N)$ is

$$\widehat{\Omega}(c, N) := \left\{(z, Z) \in \Omega \times \mathbb{C}^N \mid \|Z\|^2 < \mathcal{K}_{\Omega}(z)^{-c}\right\},$$

where $\mathbb{C}^N$ is endowed with the standard Hermitian structure.

The Cartan–Hartogs domain $\widetilde{\Omega}(\mu, N)$ is linearly equivalent to the Bergman–Hartogs domain:

$$\widetilde{\Omega}(\mu, N) \simeq \widehat{\Omega}(\mu/\gamma, N).$$

1.5. Example: Thullen domains. Let $V = \mathbb{C}^n$ be the standard Hermitian vector space, with scalar product $(z \mid t) = \sum_{j=1}^{n} z_j \overline{t_j}$ and Hermitian norm $\|z\|^2 = (z \mid z)$.

The associated symmetric domain is the Hermitian unit ball $\Omega = B_n$ of V. The genus of Ω is $g = n + 1$. The generic norm is

$$\mathcal{N}(z, t) = 1 - (z \mid t).$$

The Cartan–Hartogs domain $\widetilde{\Omega}(\mu, N)$ is then

$$\widetilde{\Omega}(\mu, N) = \left\{(z, Z) \in V \times \mathbb{C}^N \;\middle|\; \|z\|^2 + \|Z\|^{2/\mu} < 1\right\}.$$

These domains are called *Thullen domains* and also known as *complex ellipsoids*, or *complex ovals*, or *egg domains.*

Let $\Omega = B_n$ be the Hermitian unit ball of $V = \mathbb{C}^n$. For $\mu = 1$, $\widetilde{\Omega}(\mu, N)$ is the Hermitian unit ball B_{n+N} of $\mathbb{C}^{n+N}$ and is homogeneous.

P r o p o s i t i o n 1. *The Thullen domain $\widetilde{\Omega}(\mu, N)$ is biholomorphic to B_{n+N} if and only if $\mu = 1$.*

Proof. Let $f : B_{n+N} \to \widetilde{\Omega}(\mu, N)$ be a biholomorphism. By composing f with a suitable automorphism of B_{n+N}, we may assume that $f(0) = 0$. As B_{n+N} is a bounded circled domain and $\widetilde{\Omega}(\mu, N)$ is bounded, a lemma of H. Cartan implies that f is linear. It is then easy to check that the image of the boundary of B_{n+N} by f is the boundary of $\widetilde{\Omega}(\mu, N)$ if and only if $\mu = 1$. $\square$

2. Boundary and automorphisms

2.1. Strictly pseudoconvex boundary points. Let Ω be a bounded complex domain. Let $c > 0$ be a positive real number and $N > 0$ an integer. Let $X : \Omega \times \mathbb{C}^N \to (0, +\infty)$ be defined by

$$X(z, Z) := \mathcal{K}_\Omega(z)^c \, \|Z\|^2 .$$

Proposition 2. *The points of*

$$\partial_0 \widehat{\Omega}(c, N) := \left\{(z, Z) \in \Omega \times \mathbb{C}^N \mid \|Z\|^2 = \mathcal{K}_\Omega(z)^{-c}\right\}$$

are strictly pseudoconvex boundary points of $\widehat{\Omega}(c, N)$ *.*

This property has been noticed by Ahn Heungju, Byun Jisoo, Park Jong-do [1] when Ω is a bounded symmetric domain of classical type, and proved by them case-by-case for symmetric domains of classical type.

Proof. Consider the function

$$\ln X(z, Z) = c \ln \mathcal{K}_\Omega(z) + \ln \|Z\|^2 .$$

Its Levi form at (z, Z) is

$$\begin{aligned}\mathcal{L}_{(z,Z)}((w_1, W_1), (w_2, W_2)) &= \partial_{(w_1,W_1)} \overline{\partial}_{(w_2,W_2)} \ln X(z, Z) \\ &= c\, \partial_{w_1} \overline{\partial}_{w_2} \ln \mathcal{K}_\Omega(z) + \partial_{W_1} \overline{\partial}_{W_2} \ln \|Z\|^2 .\end{aligned}$$

Then $\partial_{w_1} \overline{\partial}_{w_2} \ln \mathcal{K}_\Omega(z)$ is the Bergman metric $h_z(w_1, w_2)$ of Ω at z and

$$\partial_W \overline{\partial}_W \ln \|Z\|^2 = \frac{\|Z\|^2 \|W\|^2 - |(W \mid Z)|^2}{\|Z\|^4} .$$

The complex tangent hyperplane $H_{(z,Z)}$ to $\partial_0 \widehat{\Omega}(c, N) = \{\ln X(z, Z) = 0\}$ at (z, Z) is

$$H_{(z,Z)} = \left\{(w, W) \mid c \langle \partial \ln \mathcal{K}_\Omega(z), w\rangle + \frac{(W \mid Z)}{\|Z\|^2} = 0\right\}.$$

For $(w, W) \in H_{(z,Z)}$,

$$\mathcal{L}_{(z,Z)} ((w, W), (w, W)) = h_z(w, w) + \frac{\|Z\|^2 \|W\|^2 - |(W \mid Z)|^2}{\|Z\|^4} \geqslant 0.$$

If $\mathcal{L}_{(z,Z)}((w, W), (w, W)) = 0$, then $w = 0$, which implies $(W \mid Z) = 0$, hence $\mathcal{L}_{(z,Z)}((w, W), (w, W)) = \|Z\|^{-2} \|W\|^2$ and $W = 0$. □

2.2. Automorphisms of Cartan–Hartogs domains. Let Ω be a bounded irreducible circled symmetric domain in V , with generic norm N , genus γ and Bergman kernel $K(z, t)$.

Let $\widetilde{\Omega}$ be the Cartan–Hartogs domain ($\mu > 0$, $N \geqslant 1$)

$$\widetilde{\Omega} = \widetilde{\Omega}(\mu, N) = \left\{(z, Z) \in \Omega \times \mathbb{C}^m \mid \|Z\|^2 < N(z, z)^\mu\right\}.$$

Define $X : \widetilde{\Omega} \to [0, 1)$

$$X(z, Z) = \frac{\|Z\|^2}{N(z, z)^\mu}.$$

2.2.1. *Boundary of Cartan–Hartogs domains.* The boundary of the Cartan domain Ω is a disjoint union of locally closed manifolds

$$\partial\Omega = \coprod_{j=1}^{r} \partial_j \, \Omega.$$

The boundary of the Cartan–Hartogs domain $\widetilde{\Omega} = \widetilde{\Omega}(\mu, N)$ is

$$\partial\widetilde{\Omega} = \partial_0 \, \widetilde{\Omega} \sqcup (\partial\Omega \times \{0\}) = \coprod_{j=0}^{r} \partial_j \, \widetilde{\Omega},$$

with

$$\begin{aligned} \partial_0\widetilde{\Omega} &= \left\{(z, Z) \in \Omega \times \mathbb{C}^N \;\mid\; \|Z\|^2 = N(z, z)^\mu\right\}, \\ \partial_j\widetilde{\Omega} &= \partial_j \, \Omega \times \{0\} \qquad (1 \leqslant j \leqslant r). \end{aligned}$$

The points of $\partial_0\widetilde{\Omega}$ are strictly pseudoconvex boundary points.

2.2.2. *Restricted automorphisms of Cartan–Hartogs domains.* Denote by $\operatorname{Aut}' \widetilde{\Omega}$ the subgroup of automorphisms of $\widetilde{\Omega}$ which leave X invariant.

P r o p o s i t i o n 3. *The group* $\operatorname{Aut}' \widetilde{\Omega}$ *consists of all* $\Psi : (z, Z) \mapsto (\Phi(z), \psi(z) U(Z))$, *where* $\Phi \in \operatorname{Aut} \Omega$, $U : C^N \to C^N$ *is special unitary and* ψ *satisfies*

$$|\psi(z)|^2 = \left(\frac{N(\Phi z, \Phi z)}{N(z, z)}\right)^\mu.$$

For $\Phi \in \operatorname{Aut} \Omega$, *let* $z_0 = \Phi^{-1}(0)$ *; then the functions* ψ *satisfying this condition are the functions*

$$\psi(z) = \mathrm{e}^{\mathrm{i}\,\theta} \, \frac{N\left(z_0, z_0\right)^{\mu/2}}{N\left(z, z_0\right)^\mu}.$$

The orbits of $\operatorname{Aut}' \widetilde{\Omega}$ *are the level sets* $\Sigma_\lambda = \{X = \lambda \mid \lambda \in [0, 1)\}$.

See [3].

2.2.3. *The automorphism group of a Cartan–Hartogs domain.* The following result is proved by Ahn Heungju, Byun Jisoo, Park Jong-do [1] when Ω is a symmetric domain of classical type.

Theorem 1. (1) *The Cartan–Hartogs domain* $\widetilde{\Omega}(\mu, N)$ *is homogeneous if and only if* Ω *is of type* $I_{1,n}$ *(that is, an Hermitian ball of dimension* n *) and* $\mu = 1$ *. Then* $\widetilde{\Omega}(1, N)$ *is symmetric of type* $I_{1,n+m}$ *.*
(2) *If* $\widetilde{\Omega} = \widetilde{\Omega}_m(\mu)$ *is not homogeneous, then* $\operatorname{Aut}\widetilde{\Omega} = \operatorname{Aut}'\widetilde{\Omega}$ *.*

The proof relies on the Wong–Rosay theorem:

Theorem. [2] *Let* D *be a bounded complex domain and* ξ_0 *a strictly pseudoconvex* C^2 *boundary point of* D *. If there exist an interior point* $x \in D$ *and a sequence* (T_k) *of holomorphic automorphisms of* D *, such that* $T_k(x) \to \xi_0$ *, then* D *is biholomorphic to an Hermitian ball.*

The proof of Ahn–Byun–Park relies on the strict pseudoconvexity of $\partial_0\widetilde{\Omega}(\mu, N)$, so this proof is valid for any irreducible symmetric domain Ω .

Proof. Let

$$\Phi \in \operatorname{Aut}\widetilde{\Omega}(\mu, N),$$
$$z_j \in \Omega \to \zeta \in \partial\Omega.$$

There exist

$$g_j \in \operatorname{Aut}\Omega \quad \text{such that} \quad g_j(0) = z_j,$$
$$\widetilde{g}_j \in \operatorname{Aut}\widetilde{\Omega}(\mu, N) \quad \text{such that} \quad \widetilde{g}_j(0,0) = (z_j, 0).$$

Then

$$(\Phi(z_j, 0)) = (T_j(0,0)), \qquad T_j = \Phi \circ \widetilde{g}_j \in \operatorname{Aut}\widetilde{\Omega}(\mu, N).$$

The main steps of the proof are then

- If (z_j) has a subsequence such that $(\Phi(z_j, 0))$ converges to a point $\xi_0 \in \partial_0\widetilde{\Omega}(\mu, N)$, then $\widetilde{\Omega}(\mu, N)$ is biholomorphic to an Hermitian ball by the Wong–Rosay theorem.
- $\widetilde{\Omega}(\mu, N)$ is biholomorphic to an Hermitian ball if and only if Ω is an Hermitian ball and $\mu = 1$.
- If $\widetilde{\Omega}(\mu, N)$ is not an Hermitian ball, then $\Phi(\Omega \times \{0\}) = \Omega \times \{0\}$ for all $\Phi \in \operatorname{Aut}\widetilde{\Omega}(\mu, N)$.
- Let $\Phi \in \operatorname{Aut}\widetilde{\Omega}(\mu, N)$. If $\Phi(\Omega \times \{0\}) = \Omega \times \{0\}$, then $\Phi \in \operatorname{Aut}'\widetilde{\Omega}(\mu, N)$. □

2.3. Bergman–Hartogs domains. From now on, we assume that Ω is a bounded *homogeneous* domain. Let G denote its automorphism group.

2.3.1. *Restricted automorphisms.* For $g \in G$, let $\widetilde{g} \in \operatorname{Aut}\widehat{\Omega}(c, N)$ be defined by

$$\widetilde{g}(z, Z) := (gz, Jg(z)^c Z)\,.$$

Note that the function $z \mapsto Jg(z)^c$ is in general not unique and is defined up to multiplication by a power of $\exp(2i\pi c)$. The group

$$\widetilde{G} = \{\widetilde{g} \mid \widetilde{g}(z, Z) = (gz, Jg(z)^c Z), \ g \in G\}$$

is a covering of G and a subgroup of $\operatorname{Aut} \widehat{\Omega}(c, N)$.

D e f i n i t i o n 3. The *restricted automorphism group of* $\widehat{\Omega}(c, N)$ is

$$\operatorname{Aut}' \widehat{\Omega}(c, N) = \left\{\Phi \in \operatorname{Aut} \widehat{\Omega}(c, N) \mid X \circ \Phi = X\right\},$$

where $X(z, Z) := \mathcal{K}_\Omega(z)^c \|Z\|^2$.

P r o p o s i t i o n 4. *Let* $\Phi \in \operatorname{Aut} \widehat{\Omega}(c, N)$. *The following properties are equivalent:*

1. $\Phi \in \operatorname{Aut}' \widehat{\Omega}(c, N)$;
2. $\Phi(\Omega \times \{0\}) = \Omega \times \{0\}$;
3. *there exist* $g \in G$ *and* $U \in \mathsf{U}(N)$ *such that* $\Phi(z, Z) = (gz, Jg(z)^c UZ)$.

2.3.2. *The automorphism group of a Bergman–Hartogs domain.*

Theorem 2. (1) *The Bergman–Hartogs domain* $\widehat{\Omega}(c, N)$ *is homogeneous if and only if* Ω *is an Hermitian ball of dimension* n *and* $c = \dfrac{1}{n+1}$. *Then* $\widehat{\Omega}\left(\dfrac{1}{n+1}, N\right)$ *is an Hermitian ball of dimension* $n + N$.

(2) *In all other cases,* $\operatorname{Aut} \widehat{\Omega}(c, N) = \operatorname{Aut}' \widehat{\Omega}(c, N)$.

The main steps of the proof are the same than for Cartan–Hartogs domains:

- If (z_j) has a subsequence such that $(\Phi(z_j, 0))$ converges to a point $\xi_0 \in \partial_0 \widehat{\Omega}(c, N)$, then $\widehat{\Omega}(c, N)$ is biholomorphic to an Hermitian ball by the Wong–Rosay theorem.
- $\widehat{\Omega}(c, N)$ is biholomorphic to an Hermitian ball if and only if Ω is an Hermitian ball and $c = \dfrac{1}{n+1}$.
- If $\widehat{\Omega}(c, N)$ is not an Hermitian ball, then $\Phi(\Omega \times \{0\}) = \Omega \times \{0\}$ for all $\Phi \in \operatorname{Aut} \widehat{\Omega}(c, N)$.
- Let $\Phi \in \operatorname{Aut} \widehat{\Omega}(c, N)$. If $\Phi(\Omega \times \{0\}) = \Omega \times \{0\}$, then $\Phi \in \operatorname{Aut}' \widehat{\Omega}(c, N)$. □

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